ON IMPROVING THE ESTIMATES OF THE SOLUTIONS OF THE SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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There exist many different variants of sufficient criteria for the stability of motion. Estimates of the solutions of the differential equations describing the disturbed motion of the mechanical systems investigated, are taken as a principle in these criteria. The accuracy of these estimates depends essentially on the choice of coefficients of a specified quadratic form. With the aid of this form, an auxiliary function for the study of the stability of motion is derived.

In the present paper we compare various formulás for estimates. Some methods of improving auxiliary functions are suggested so as to enable obtaining sufficiently flexible and accurate estimates for the solution of differential equations with variable coefficients.

1. We will consider the system of homogeneous differential equations

$$\frac{dx_s}{dt} = \sum_{m=1}^{n} a_{sm}(t)\, x_m \qquad (s = 1, \ldots, n) \tag{1.1}$$

where $a_{ss}(t)$ are continuous, differentiable functions on the given finite time interval $[t_0, T]$. Many different methods of forming estimates for the solutions of such a system of differential equations can be found in the literature. We will try to find a way of improving these estimates.

In order to study the stability of system (1.1) it is usual to introduce the function

$$V(t, x) = e^{\gamma(t)} A(t, x), \qquad A(t, x) = \sum_{s=1}^{n}\sum_{m=1}^{n} A_{sm}(t)\, x_s x_m, \qquad (A_{sm}(t) = A_{ms}(t)) \tag{1.2}$$

where $A(t, x)$ is a quadratic form in the variables $x_1, \ldots, x_n$ constructed with the aid of the coefficient matrix of the specified system of differential equations

$$a(t) = \left\| \begin{matrix} a_{11}(t) & \ldots & a_{1n}(t) \\ \ldots & \ldots & \ldots \\ a_{n1}(t) & \ldots & a_{nn}(t) \end{matrix} \right\| \tag{1.3}$$

The time derivative of function (1.2), obtained by virtue of Equations (1.1)

$$V^{\cdot}(t, x) = e^{\gamma(t)} B(t, x), \quad B(t, x) = \sum_{s=1}^{n} \sum_{m=1}^{n} B_{sm}(t) x_s x_m \qquad (B_{sm}(t) = B_{ms}(t)) \quad (1.4)$$

will also be a quadratic form in the same variables $x_1, \ldots, x_n$ and

$$B_{sm} = B_{ms} = A^{\cdot}_{sm} + \gamma^{\cdot} A_{sm} + \sum_{k=1}^{n} (a_{ks} A_{km} + a_{km} A_{ks}) \qquad (s, m = 1, \ldots, n) \quad (1.5)$$

We will subject the coefficients of the quadratic forms (1.2) and (1.4) to the conditions

$$A_k(t) > 0 \quad (k = 1, \ldots, n) \qquad (-1)^k B_k(t) > 0 \quad (k = 1, \ldots, n) \qquad (1.6)$$

$$A_k(t) = \begin{vmatrix} A_{11}(t) & \ldots & A_{1k}(t) \\ \ldots & \ldots & \ldots \\ A_{k1}(t) & \ldots & A_{kk}(t) \end{vmatrix} (k=1,\ldots,n), \qquad B_k(t) = \begin{vmatrix} B_{11}(t) & \ldots & B_{1k}(t) \\ \ldots & \ldots & \ldots \\ B_{k1}(t) & \ldots & B_{kk}(t) \end{vmatrix} (k=1,\ldots,n)$$

The it can be asserted that the introduced auxiliary function (1.2) is positive sign-definite and that its time derivative (1.4) is negative sign-definite. Then, obviously, we have

$$V(t, x) \leqslant V(t_0, x_0), \quad \text{for} \quad A(t, x) \leqslant A(t_0, x_0) \exp\left(-\int_{t_0}^{t} \gamma^{\cdot}(\tau)\, d\tau\right) \qquad (1.7)$$

The validity of the following inequality may be shown (see [1], p.38)

$$\frac{A_n(t)}{M_k(t)} x_k^2 \leqslant A(t, x), \qquad A_n(t) = \begin{vmatrix} A_{11}(t) & \ldots & A_{1n}(t) \\ \ldots & \ldots & \ldots \\ A_{n1}(t) & \ldots & A_{nn}(t) \end{vmatrix} \qquad (1.8)$$

where $A_n(t)$ is the discriminant of the quadratic form $A(t, x)$, and $M_k(t)$ is the minor corresponding to the element in the kth row and kth column.

Then we obtain

$$|x_k(t)| \leqslant \left(A(t_0, x_0) \frac{M_k(t)}{A_n(t)}\right)^{1/2} \exp\left(-\frac{1}{2}\int_{t_0}^{t} \gamma^{\cdot}(\tau)\, d\tau\right) \qquad (k = 1, \ldots, n) \quad (1.9)$$

This formula is one of the possible estimates from above of the modulus of the particular solution of system (1.1) corresponding to the initial conditions $x_{10}, \ldots, x_{n0}$. If in inequality (1.9) the number $A(t_0, x_0)$ defined by (1.2) at time $t = t_0$ is replaced by the upper bound

$$A^+(t_0, x_0^*) = \sum_{s=1}^{n} \sum_{m=1}^{n} |A_{sm}(t_0)|\, x_{s0}^* x_{m0}^* \qquad (1.10)$$

then we obtain the following upper bound for the modulus of the general solution of system (1.1):

$$|x_k(t)| \leqslant \left(A^+(t_0, x_0^*) \frac{M_k(t)}{A_n(t)}\right)^{1/2} \exp\left(-\frac{1}{2}\int_{t_0}^{t} \gamma^{\cdot}(\tau)\, d\tau\right) \qquad (k = 1, \ldots, n) \quad (1.11)$$

corresponding to arbitrary initial conditions subjected to the sole condition

$$|x_{10}| \leqslant x_{10}^*, \ldots, |x_{n0}| \leqslant x_{n0}^* \qquad (1.12)$$

where x_{k0}^* $(k = 1, \ldots, n)$ are certain fixed numbers.

Apart from the estimates (1.9) and (1.11), we will find it useful in the following to use estimates based on a regular family of quadratic forms.

Assuming $\gamma(t) \equiv 0$ in (1.2), and consequently

$$V = A(t, x) = \sum_{s=1}^{n} \sum_{m=1}^{n} A_{sm}(t) x_s x_m \qquad (1.13)$$

The time derivative of this form by virtue of the differential equations (1.1) is

$$A^{\cdot}(t, x) = B^{+}(t, x) = \sum_{s=1}^{n} \sum_{m=1}^{n} B_{sm}^{+}(t) x_s x_m \qquad (1.14)$$

thereby

$$B_{sm}^{+} = B_{ms}^{+} = A^{\cdot}_{sm} + \sum_{k=1}^{n} (a_{ks} A_{km} + a_{km} A_{ks}) \qquad (s, m = 1, \ldots, n) \qquad (1.15)$$

On the assumption that the quadratic form (1.13) is positive definite, we can combine the quadratic forms (1.13) and (1.14) to yield the regular family

$$B^{+} - \mu A = \sum_{s=1}^{n} \sum_{m=1}^{n} (B_{sm} - \mu A_{sm}) x_s x_m \qquad (1.16)$$

Let $\mu_1(t)$, ..., $\mu_n(t)$ be its characteristic values, i.e. the roots of Equation

$$(-1)^n \begin{vmatrix} B_{11}^{+} - \mu A_{11} \ldots B_{1n}^{+} - \mu A_{1n} \\ \ldots\ldots\ldots\ldots \\ B_{n1}^{+} - \mu A_{n1} \ldots B_{nn}^{+} - \mu A_{nn} \end{vmatrix} = 0 \qquad (1.17)$$

then the following inequality is valid

$$\mu_{-}(t) \leqslant \frac{A^{\cdot}(t, x)}{A(t, x)} \leqslant \mu_{+}(t) \qquad (1.18)$$

where

$$\mu_{-}(t) = \min[\mu_1(t), \ldots, \mu_n(t)], \ \mu_{+}(t) = \max[\mu_1(t), \ldots, \mu_n(t)] \qquad (1.19)$$

By integrating inequality (1.18) over the interval $[t_0, T]$ and making use of relations (1.8), (1.10) and (1.12), we obtain an estimate of the general solution of system (1.1) with initial values lying in the parallelepiped (1.12):

$$|x_k(t)| \leqslant \left(A^{+}(t_0, x_0^{*}) \frac{M_k(t)}{A_n(t)} \right)^{1/2} \exp\left(\frac{1}{2} \int_{t_0}^{t} \mu_{+}(\tau)\, d\tau \right) \qquad (k = 1, \ldots, n) \qquad (1.20)$$

Other estimates (see, for example, [1]) are either obtained by replacing the quadratic forms (1.2) and (1.13) by the particular types

$$v(t, x) = e^{\gamma(t)} \sum_{m=1}^{n} \alpha_m(t) x_m^2, \qquad \alpha(t, x) = \sum_{m=1}^{n} \alpha_m(t) x_m^2 \qquad (1.21)$$

or they are consequences of the preceding results. Therefore, these estimates will, in general, be poorer than estimates (1.11) and (1.20).

It is easily proved that estimate (1.20) is more accurate than estimate (1.11). In fact, since the quadratic form (1.13) is sign-definite, the pair of quadratic forms (1.13) and (1.4) can always be reduced to the cononical form

$$B^{+}(t, x) = C(t, y) = \sum_{m=1}^{n} \mu_m(t) y_m^2, \qquad A(t, x) = D(t, y) = \sum_{m=1}^{n} y_m^2 \qquad (1.22)$$

if we treat the time t as a parameter. In this case, however, the quadratic forms (1.2) and (1.4) assume the representation

$$V(t, x) = e^{\gamma(t)} \sum_{m=1}^{n} y_m^2, \qquad V^{\cdot}(t, x) = e^{\gamma(t)} \sum_{m=1}^{n} [\mu_m(t) + \gamma^{\cdot}(t)] y_m^2 \qquad (1.23)$$

respectively.

The fact that (1.6) is sign-definite yields the inequalities

$$\mu_m(t) < -\gamma^{\cdot}(t) \qquad (m = 1, \ldots, n) \tag{1.24}$$

from which follows the inequality

$$\mu_+(t) < -\gamma^{\cdot}(t) \tag{1.25}$$

Since the right-hand sides of inequalities (1.11) and (1.20) differ only in the arguments of the exponental function, it follows that estimate (1.20) is more accurate than estimate (1.11). We will therefore use only estimate (1.20) in what follows.

2. The accuracy of the above estimates of the solutions of the system of equations (1.1) depends not only on the way of obtaining these estimates but also on the quadratic form used as the Liapunov function

$$V(t, x) = \sum_{s=1}^{n} \sum_{m=1}^{n} A_{sm}(t) x_s x_m \qquad (A_{sm}(t) = A_{ms}(t)) \tag{2.1}$$

This function is usually obtained proceeding from the well-known partial differential equation [2 and 3]

$$\sum_{s=1}^{n} \frac{\partial V}{\partial x_s} (a_{s1}(t) x_1 + \ldots + a_{sn}(t) x_n) = U(t, x) = \sum_{s=1}^{n} u_{ss}(t) x_s^2 \tag{2.2}$$

where $U(t, x)$ is a given negative-definite quadratic form.

Now we will consider some ways of choosing the coefficients of form $U(t, x)$ which may be useful for the solution of concrete problems.

F i r s t m e t h o d . In determining the form (2.1) it is usual to assume that all the coefficients of the specified qudratic form $U(t, x)$ are constant and equal to -1 .

This often leads to a coarsening of the obtained estimate.

S e c o n d m e t h o d . Having represented the corfficients of the quadratic form (2.1) as

$$A_{sm}(t) = A_{ms}(t) = \sum_{k=1}^{n} d_{sm}^{kk}\ t)\, u_{kk}(t) \qquad (s,\ m = 1, \ldots, n) \tag{2.3}$$

where $d_{sm}^{kk}(t)$ $(s, m, k = 1, \ldots, n,\ m \geqslant s)$ are functions defined in terms of the coefficients of the given system (1.1) (see [1], Chapt.II, Section 3), we will choose the coefficients $u_{ss}(t)$ of the specified quadratic form $U(t, x)$ from the condition

$$\frac{1}{A_{11}(t)}\left(\frac{dA_{11}}{dt} + u_{11}(t)\right) = \ldots = \frac{1}{A_{nn}(t)}\left(\frac{dA_{nn}}{dt} + u_{nn}(t)\right) = \mu(t) \tag{2.4}$$

hereby, out of all the possible values of $\mu(t)$, the smallest possible is chosen for which the form (2.1) is positive definite.

Substituting Expressions (2.3) for the coefficients into Equation (2.4), we obtain the following system of linear differential equations:

$$\sum_{k=1}^{n} d_{ii}^{kk}(t) \frac{du_{kk}}{dt} = \sum_{k=1}^{n} \left(\mu(t) d_{ii}^{kk}(t) - \frac{dd_{ii}^{kk}}{dt}\right) u_{kk} - u_{ii} \qquad (i = 1, \ldots, n) \tag{2.5}$$

The solution of this system will be sought by means of the following iterative method:

$$\mu(t) = \lim_{l\to\infty} \mu^{(l)}(t), \qquad u_{kk}(t) = \lim_{l\to\infty} u_{kk}^{(l)}(t) \qquad (k = 1, \ldots, n) \tag{2.6}$$

where $\mu^{(l)}(t)$, and $u^{(l)}{}_{kk}(t)$ satisfy the system of algebraic equations

$$\sum_{k=1}^{n} \left(\mu^{(0)}(t) d_{ii}^{kk}(t) - \frac{dd_{ii}^{kk}}{dt}\right) u_{kk}{}^{(0)}(t) - u_{ii}{}^{(0)}(t) = 0 \qquad (i = 1, \ldots, n) \tag{2.7}$$

$$\sum_{k=1}^{n}\left(\mu^{(l)}(t)\,d_{ii}^{kk}(t)-\frac{dd_{ii}^{kk}}{dt}+d_{ii}^{kk}(t)\,\frac{du_{kk}^{(l-1)}/dt}{u_{kk}^{(l-1)}(t)}\right)u_{kk}^{(l)}(t)-u_{ii}^{(l)}(t)=0$$

$$(i=1,\ldots,n;\ l=1,\ldots) \tag{2.8}$$

The solution of (2.7) and (2.8) in the present case, is reduced to the determination of the least possible eigenvalue for which the form (2.1) is positive definite, and of the corresponding eigenvector.

We will not study the question of convergence of the iterative process (2.6). We simply remark that the number of iterations m that is necessary in order to obtain the functions $\mu(t)$ and $u_{kk}(t)\ (k=1,\ldots,n)$ with a prescribed accuracy ε_μ and $\varepsilon_k\ (k=1,\ldots,n)$, is determined from the conditions

$$|\mu^{(m)}(t)-\mu^{(m-1)}(t)|<\varepsilon_\mu \qquad |u_{kk}^{(m)}(t)-u_{kk}^{(m-1)}(t)|<\varepsilon_k \quad (k=1,\ldots,n) \quad (t_0\leqslant t\leqslant T)$$

In many practical problems it is sufficient to limit oneself to the zeroth approximation of the iterative process, which corresponds to the choice of the coefficients $A_{ss}(t)$ and $u_{ss}(t)$ $(s=1,\ldots,n)$ from the conditions

$$\frac{1}{A_{11}(t)}\left(\frac{d^*A_{11}}{dt}+u_{11}(t)\right)=\ldots=\frac{1}{A_{nn}(t)}\left(\frac{d^*A_{nn}}{dt}+u_{nn}(t)\right)=\mu(t) \tag{2.9}$$

where d^*A_{11}/dt is the time derivative of A_{11} calculated on the assumption that the functions $u_{ss}(t)$ $(s=1,\ldots,n)$ are constant.

The practical realization of the above method consists either in finding the smallest possible eigenvalue of system (2.7) and the corresponding eigenvector on an electronic digital computer, or in selecting the coefficients u_{ss} $(s=1,\ldots,n)$ in accordance with (2.9) by simply using a slide-rule.

T h i r d m e t h o d . This method consists in selecting the negative functions $u_{ss}(t)$ $(s=1,\ldots,n)$ such that the function $\mu_+(t)$ (see (1.19)) assumes a minimum at an arbitrary instant of time in the considered interval $[t_0, T]$; thereby $\mu_1(t),\ldots,\mu_n(t)$ are the eigenvalues of the family of quadratic forms

$$B(t,x)-\mu(t)A(t,x)=\sum_{s=1}^{n}\sum_{m=1}^{n}\left(\frac{dA_{sm}}{dt}+\delta_{sm}u_{sm}(t)-\mu(t)A_{sm}(t)\right)x_sx_m \tag{2.10}$$

(δ_{sm} is the Kronecker symbol)

The realization of this method involves a preliminary calculation of the functions $d_{sm}^{kk}(t)$ $(s,m,k=1,\ldots,n$ and $s\leqslant m)$ and the application of the Monte Carlo method [4] to the choice of variable functions $u_{ss}(t)$ $(s=1,\ldots,n)$.

If the specified quadratic form $U(t,x)$ is assumed to be negative sign-definite but not in the canonical form, i.e.

$$U(t,x)=\sum_{s=1}^{n}\sum_{m=1}^{n}u_{sm}(t)\,x_sx_m \tag{2.11}$$

then more flexible estimates can be obtained. However, such an improvement of the estimates does not always turn out to be expedient, since it involves a considerable increase in the extent of the computational operations and additional expenditure of machine time.

3. As an example we will consider the system of differential equations

$$\frac{dx_1}{dt}=a_{11}(t)x_1+x_2,\qquad \frac{dx_2}{dt}=a_{21}(t)x_1+a_{22}(t)x_2 \tag{3.1}$$

specified on the closed interval [0,2]. The values of the coefficients are

$t=$	0	0.4	0.8	1.2	1.6	2.0
$a_{11}=$	—4.40	—4.42	—4.44	—4.46	—4.48	—4.50
$a_{21}=$	—131.50	—131.66	—131.83	—131.99	—132.15	—132.30
$a_{22}=$	—6.300	—6.325	—6.342	—6.362	—6.382	—6.400

Let the initial conditions lie in the rectangle

$$|x_{10}| \leqslant 0.01, \qquad |x_{20}| \leqslant 0.5 \tag{3.2}$$

The estimates of the general solution of system (3.1) will be constructedd according to Formula (1.20), which in the present case assumes the form

$$|x_1(t)| \leqslant X_1(t) = \left(\left[\frac{A_{11}(0)}{10000} + \frac{|A_{12}(0)|}{100} + \frac{A_{22}(0)}{4}\right] \frac{A_{22}(t)}{A_{11}(t)A_{22}(t) - A_{12}^2(t)}\right)^{1/2} \times$$

$$\times \exp\left(\frac{1}{2}\int_0^t \mu_+(\tau)\,d\tau\right) \tag{3.3}$$

$$|x_2(t)| \leqslant X_2(t) = \left(\left[\frac{A_{11}(0)}{10000} + \frac{|A_{12}(0)|}{100} + \frac{A_{22}(0)}{4}\right] \frac{A_{11}(t)}{A_{11}(t)A_{22}(t) - A_{12}^2(t)}\right)^{1/2} \times$$

$$\times \exp\left(\frac{1}{2}\int_0^t \mu_+(\tau)\,d\tau\right) \tag{3.4}$$

We have considered two variants of the coefficients

$$u_{11}(t) = -1, \quad u_{22} = -1, \quad u_{11}(t) = -1.421, \quad u_{22}(t) = -0.01 \tag{3.5}$$

of the quadratic form

$$U(t, x) = u_{11}(t)x_1^2 + u_{22}(t)x_2^2 \tag{3.6}$$

Here, the second set of coefficients $u_{11}(t)$ was obtained by the second method in the variant (2.9).

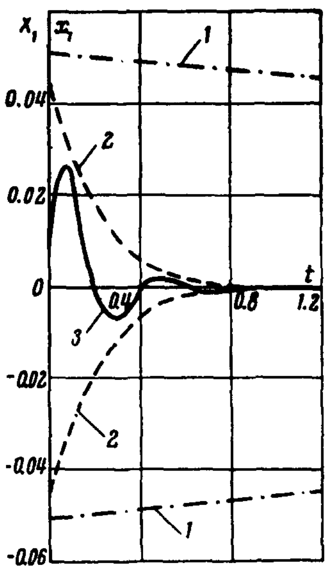

Fig. 1

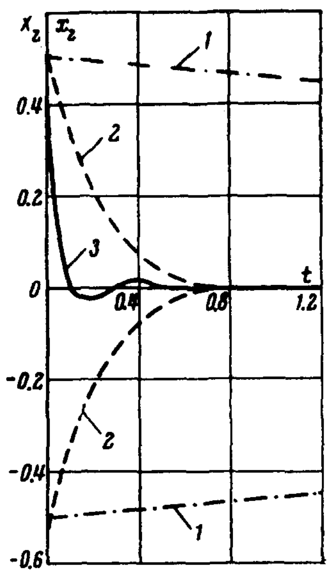

Fig. 2

Below we quote the numerical values of the function $\mu_+(t)$, together with $X_1(t)$ and $X_2(t)$ which are the estimates for the general solution of system (3.1) with initial conditions lying in the rectangle (3.2), for the two sets of coefficients:

first variant

$t =$	0	0.4	0.8	1.2	1.6	2.0
$\mu_+ =$	−0.20503	−0.20344	−0.20506	−0.20571	−0.20604	−0.20676
$X_1 =$	$0.51107 \cdot 10^{-1}$	$0.49085 \cdot 10^{-1}$	$0.47137 \cdot 10^{-1}$	$0.45263 \cdot 10^{-1}$	$0.43630 \cdot 10^{-1}$	$0.42032 \cdot 10^{-1}$
$X_2 =$	0.50438	0.48426	0.46493	0.44630	0.42996	0.41417

second variant

$t =$	0	0.4	0.8	1.2	1.6	2.0
$\mu_+ =$	−9.82016	−9.86708	−9.90771	−9.94295	−9.98739	−10.02175
$X_1 =$	$0.44075 \cdot 10^{-1}$	$0.61518 \cdot 10^{-2}$	$0.85076 \cdot 10^{-2}$	$0.11680 \cdot 10^{-3}$	$0.15912 \cdot 10^{-4}$	$0.21496 \cdot 10^{-5}$
$X_2 =$	0.52533	$0.73374 \cdot 10^{-1}$	$0.10153 \cdot 10^{-1}$	$0.13946 \cdot 10^{-2}$	$0.19080 \cdot 10^{-3}$	$0.25695 \cdot 10^{-4}$

Figs. 1 and 2 show the graphs of the estimates $X_1(t)$, $X_2(t)$ for the two sets of coefficients (3.5) and the particular solutions $x_1(t)$, $x_2(t)$ of system (3.1) for the initial conditions

$$x_{10} = 0.01, \qquad x_{20} = 0.5 \tag{3.7}$$

The numbers 1 and 2 refer to the estimates obtained by the first and second methods, respectively, and 3 refers to the particular solution.

From these graphs it is clear that, in comparison with the first method, the second one enables a considerable improvement in the quality of the estimates obtained.

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